PROBABILITY CONTENT OF ERROR ELLIPSE AND ERROR CONTOUR

(navell-08.mcd) 2006.12.02

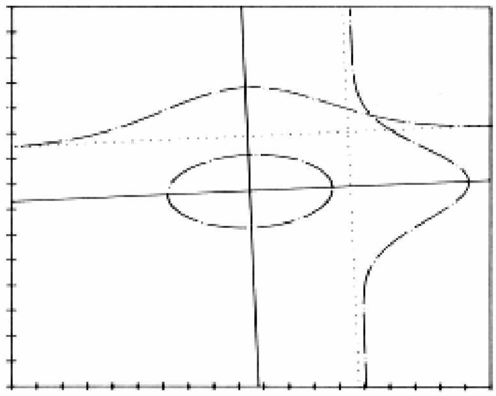
Conditions of Use, Disclaimer

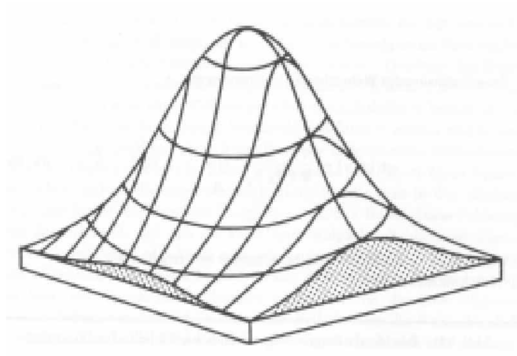
This document contains scientific work. I cannot exclude, that the algorithm or the calculations contain errors. I expressly exclude any form of liability for dammage which might arrise in practical applications by the use of this algorithm. The content of this document may only be used for private, non-commercial purposes.

Ingo Harre, Bremen, January 2001/December 2006 e-mail: Ingo.Harre@mar-it.de

NAVELL

This Mathcad sheet belongs to the NavGen suite of analysis tools for the determination of the accuracy of position determination. It provides results of numerical integration of the two-dimensional probability density function of position fixes over certain error contours.





Input data are the probability density functions in the two orthogonal axes with assumed Gaussian shape, characterised by the parameters σ_x and σ_y (left figure above) . The pair of orthogonal error distributions determines the 3D CDF, shown in the right figure. In this document the 2D PDF is integrated over specific error contours, i.e. error ellipses (left figure) and error circles, yielding the 'confidence content' or 'error probability' related to the error contour under examination.

For each error contour to be examined, five different ratios of the underlying standard deviations σ_y / σ_x are applied to show the variation of confidence content with the ellipticity of the 2D error distribution.

Input Standard Deviations

$i := 0..4$

Standard deviations of position coordinates

$\sigma_x := 1$

$\sigma_{y_i} := k_i \cdot \sigma_x$

$$k := \begin{pmatrix} 0.01 \\ 0.25 \\ 0.5 \\ 0.75 \\ 1 \end{pmatrix}$$

Ratio of SDs

Genuine Error Ellipse

Semi major axis $\sigma_x := 1$ *Semi minor axis* $\sigma_y = \begin{pmatrix} 0.010 \\ 0.250 \\ 0.500 \\ 0.750 \\ 1.000 \end{pmatrix}$

related dRMS error $dRMS_i := \sqrt{\left[\sigma_x^2 + \left(\sigma_{y_i}\right)^2\right]}$ $dRMS = \begin{pmatrix} 1.000 \\ 1.031 \\ 1.118 \\ 1.250 \\ 1.414 \end{pmatrix}$

1a. Integration Contour: Single Standard Deviations Error Ellipse

(The integration contour is the same as the genuine error ellipse)

Semi axes $a_i := \sigma_x$ $b_i := \sigma_{y_i}$

Case displayed $i := 3$ *Excentricity:* $k_3 = 0.750$

Definition of plot variables

$j := 0 .. 100$

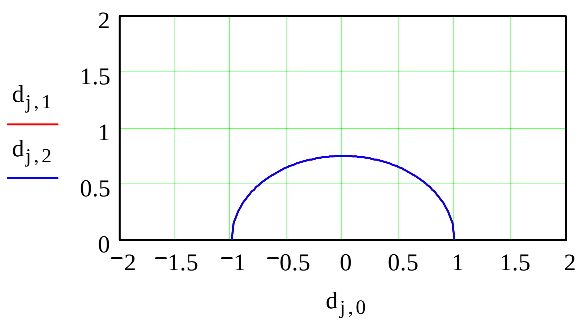
$d_{j,0} := \frac{j-50}{50}$

$d_{j,1} := \sigma_{y_i} \cdot \sqrt{1 - \frac{\left(d_{j,0}\right)^2}{\left(\sigma_x\right)^2}}$ *Error ellipse* $d_{50,0} = 0.000$ $\mathrm{Re}\left(d_{50,1}\right) = 0.750$

$d_{100,0} = 1.000$ $\mathrm{Re}\left(d_{100,1}\right) = 0.000$

$d_{j,2} := \sigma_{y_i} \cdot \sqrt{1 - \frac{\left(d_{j,0}\right)^2}{\left(\sigma_x\right)^2}}$ *Integration contour (error ellipse)* $d_{50,0} = 0.000$ $\mathrm{Re}\left(d_{50,2}\right) = 0.750$

$d_{100,0} = 1.000$ $\mathrm{Re}\left(d_{100,2}\right) = 0.000$



(Error ellipse and integration contour coincide)

Probability integral

Integration boundaries

$i := 0..4$

$a_i := \sigma_x$

$b_i := \sigma_{y_i}$

$a_i =$	$b_i =$
1.000	0.010
1.000	0.250
1.000	0.500
1.000	0.750
1.000	1.000

$$P_i := \frac{1}{2\cdot\pi\cdot\sigma_x\cdot\sigma_{y_i}}\cdot\int_{-a_i}^{a_i}\int_{-\left[\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}\right]}^{\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}} e^{\left[\left(-\frac{1}{2}\right)\cdot\left[\frac{x^2}{\sigma_x^{\,2}}+\frac{y^2}{\left(\sigma_{y_i}\right)^2}\right]\right]}\,dy\,dx$$

$k_i =$	$dRMS_i =$	$P_i =$
0.010	1.000	0.393
0.250	1.031	0.393
0.500	1.118	0.393
0.750	1.250	0.393
1.000	1.414	0.393

***Result:** the confidence content (or error probability) is 0.39. This means the probability that the position lies within the error contour is 39%. This value is independent of the ratio of the underlying standard deviations.*

The confidence content of the error ellipse can be calculated more easily by the equivalent Rayleigh cumulative probability function (CDF).

with $\sigma := \sqrt{\sigma_x^{\,2}+\left(\sigma_{y_4}\right)^2}$ *and* $R := \sigma$ $\qquad \sigma = 1.414$

$$P(R) := 1 - \exp\left(\frac{-R^2}{2\cdot\sigma^2}\right) \qquad P(R) = 0.393$$

1b. Integration Contour: Double Standard Deviations Error Ellipse

Semi axes $a_i := \sigma_x$ $b_i := \sigma_{y_i}$

Case displayed $i := 3$ *Excentricity:* $k_3 = 0.750$

Definition of plot variables

$j := 0..100$

$$d_{j,0} := \frac{j-50}{25}$$

$$d_{j,1} := \left(\sigma_y\right)_i \cdot \sqrt{1 - \frac{\left(d_{j,0}\right)^2}{\left(\sigma_x\right)^2}}$$

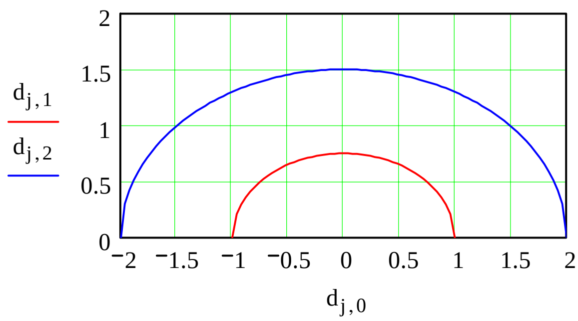
Error ellipse $d_{50,0} = 0.000$ $\mathrm{Re}\left(d_{50,1}\right) = 0.750$

$d_{100,0} = 2.000$ $\mathrm{Re}\left(d_{100,1}\right) = 0.000$

$$d_{j,2} := \left(2\cdot\sigma_y\right)_i \cdot \sqrt{1 - \frac{\left(d_{j,0}\right)^2}{\left(2\cdot\sigma_x\right)^2}}$$

Integration contour (error ellipse) $d_{50,0} = 0.000$ $\mathrm{Re}\left(d_{50,2}\right) = 0.750$

$d_{100,0} = 2.000$ $\mathrm{Re}\left(d_{100,2}\right) = 0.000$



solid red line: genuine error ellipse

dashed blue line: integration contour

Probability integral

Integration boundaries

$i := 0..4$

$a_i := 2\cdot\sigma_x$

$b_i := 2\cdot\sigma_{y_i}$

$a_i =$	$b_i =$
2.000	0.020
2.000	0.500
2.000	1.000
2.000	1.500
2.000	2.000

$$P_i := \frac{1}{2\cdot\pi\cdot\sigma_x\cdot\sigma_{y_i}}\cdot\int_{-a_i}^{a_i}\int_{-\left[\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}\right]}^{\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}} e^{\left[\left(-\frac{1}{2}\right)\cdot\left[\frac{x^2}{\sigma_x^2}+\frac{y^2}{(\sigma_{y_i})^2}\right]\right]}\,dy\,dx$$

$k_i =$	$P_i =$
0.010	0.865
0.250	0.865
0.500	0.865
0.750	0.865
1.000	0.865

Result: *the confidence content (or error probability) is 0.865. This means the probability that the position lies within the error contour is 86.5%. This value is independent of the ratio of the underlying standard deviations.*

The confidence content of the error ellipse can be calculated more easily by the equivalent Rayleigh cumulative probability function (CDF).

with $\sigma := \sqrt{\sigma_x^2 + (\sigma_{y_4})^2}$ *and* $R := 2\sigma$

$$P(R) := 1 - \exp\left(\frac{-R^2}{2\cdot\sigma^2}\right)$$

$P(R) = 0.865$

1c. Integration Contour: 95% - Error Ellipse

Probability integral

Integration boundaries

$i := 0..4$

$a_i := 2.449\cdot\sigma_x$

$b_i := 2.449\cdot\sigma_{y_i}$

Iteratively produced Sigma-multiplier for p=0,95 integration boundaries.

$a_i =$

2.449
2.449
2.449
2.449
2.449

$b_i =$

0.024
0.612
1.224
1.837
2.449

$$P_i := \frac{1}{2\cdot\pi\cdot\sigma_x\cdot\sigma_{y_i}}\cdot\int_{-a_i}^{a_i}\int_{-\left[\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}\right]}^{\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}} e^{\left[\left(-\frac{1}{2}\right)\cdot\left[\frac{x^2}{\sigma_x^{\,2}}+\frac{y^2}{(\sigma_{y_i})^2}\right]\right]}\,dy\,dx$$

$k_i =$

0.010
0.250
0.500
0.750
1.000

$P_i =$

0.950
0.950
0.950
0.950
0.950

Result: *the confidence content (or error probability) chosen is 0.95. This means the probability that the position lies within the error contour is 95%. This value is independent of the ratio of the underlying standard deviations.*

The respective multiplier for both standard deviations is 2.449

This value is used in NavGen to produce the 95% error ellipse shown in the diagram of the individual position fixes.

The multiplier for the standard deviations yielding an error ellipse with a confidence content of 95 % can be calculated more easily by resolving the Raleigh CDF for the radius R:

$P(R) := 0.95 \qquad \sigma := 1$

$R := \sqrt{-2\cdot\ln(1-P(R))}\cdot\sigma$

$R = 2.448$

2a. Integration Contour: Circle dRMS

Distance Root Mean Square (dRMS) is a commonly used error measure for position fixes, as it is quite easy to calculate. It replaces two-dimensional statistics by one-dimensional, and in fact defines an error circle, no matter what the ratio of the underlying standard deviations is. The calculation below shows that the confidence content of the related error circle is dependent on the ratio of the standard deviations. Hence, caution must prevail if the error distribution is not circular, i.e. not having identical error distributions.

$i := 0..4$

$\sigma_x := 1$

$\sigma_{y_i} := k_i \cdot \sigma_x$

$$k = \begin{pmatrix} 0.010 \\ 0.250 \\ 0.500 \\ 0.750 \\ 1.000 \end{pmatrix}$$

Radius of error circle $\quad \mathrm{dRMS}_i := \sqrt{\left[\sigma_x^{\,2} + \left(\sigma_{y_i}\right)^2\right]}$

$$\mathrm{dRMS} = \begin{pmatrix} 1.000 \\ 1.031 \\ 1.118 \\ 1.250 \\ 1.414 \end{pmatrix}$$

Case displayed $\quad i := 2 \quad$ *axis ratio:*

$j := 0..1000$

$k_i = 0.500$

$\sigma_x = 1.000$

$\sigma_{y_i} = 0.500$

$\mathrm{dRMS}_i = 1.118$

Definition of plot variables

$d_{j,0} := \dfrac{j - 500}{250} \qquad \max\left(d^{\langle 0 \rangle}\right) = 2.000$

$d_{j,1} := \sigma_{y_i} \cdot \sqrt{1 - \dfrac{\left(d_{j,0}\right)^2}{\left(\sigma_x\right)^2}}$ *Error Ellipse*

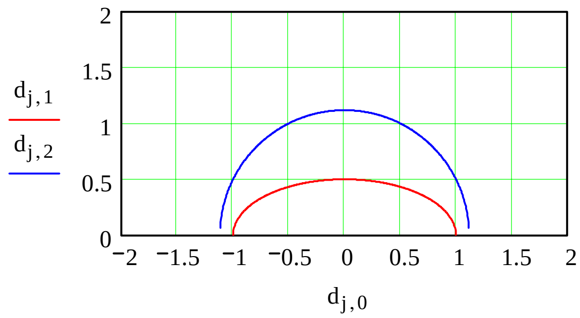
$d_{500,0} = 0.000 \qquad \mathrm{Re}\left(d_{500,1}\right) = 0.500$

$d_{1000,0} = 2.000 \qquad \mathrm{Re}\left(d_{1000,1}\right) = 0.000$

$d_{j,2} := \sqrt{\left(\mathrm{dRMS}_i\right)^2 - \left(d_{j,0}\right)^2}$ *Integration contour (error circle)*

$d_{500,0} = 0.000 \qquad \mathrm{Re}\left(d_{500,2}\right) = 1.118$

$d_{800,0} = 1.200 \qquad \mathrm{Re}\left(d_{800,2}\right) = 0.000$



solid red line:
genuine error ellipse

dashed blue line:
integration contour

Probability integral

Integration boundaries

$i := 0..4$

$a_i := dRMS_i$

$b_i := dRMS_i$

$a_i =$	$b_i =$
1.000	1.000
1.031	1.031
1.118	1.118
1.250	1.250
1.414	1.414

$$P_i := \frac{1}{2 \cdot \pi \cdot \sigma_x \cdot \sigma_{y_i}} \cdot \int_{-a_i}^{a_i} \int_{-\sqrt{(b_i)^2 \cdot \left|1 - \frac{x^2}{(a_i)^2}\right|}}^{\sqrt{(b_i)^2 \cdot \left|1 - \frac{x^2}{(a_i)^2}\right|}} e^{\left[-\frac{1}{2} \cdot \left[\frac{x^2}{\sigma_x{}^2} + \frac{y^2}{(\sigma_{y_i})^2}\right]\right]} \, dy \, dx$$

$k_i =$	$dRMS_i =$	$P_i =$
0.010	1.000	0.683
0.250	1.031	0.682
0.500	1.118	0.663
0.750	1.250	0.639
1.000	1.414	0.632

Result: *the confidence content (or error probability) varies with the ratio of the underlying standard deviations betwen 0.632 and 0.683. The dRMS error measure does not possess a fixed probability. For equal standard deviations a probability of 63.2 % applies.*

As the probability content of the dRMS error circle varies with the ratio of the underlying SDs, the Rayleigh distribution cannot be used to determine the probability content. The author has developed the following approximation for the probability content as a function of the SD ratio k:

$$Pa_i := \frac{0.6300358 \cdot 20.672132 + 0.68259309 \cdot \left(k_i\right)^{(-5.1208746)}}{20.672132 + \left(k_i\right)^{(-5.1208746)}}$$

$$2dRMS = \begin{pmatrix} 2.000 \\ 2.062 \\ 2.236 \\ 2.500 \\ 2.828 \end{pmatrix} \quad P = \begin{pmatrix} 0.683 \\ 0.682 \\ 0.663 \\ 0.639 \\ 0.632 \end{pmatrix} \quad P_a = \begin{pmatrix} 0.683 \\ 0.682 \\ 0.663 \\ 0.639 \\ 0.632 \end{pmatrix}$$

2b. Integration Contour: Circle 2dRMS

$i := 0 .. 4$

Case displayed $i := 2$ *Excentricity:* $k_i = 0.500$

Definition of plot variables

$j := 0 .. 6000$

$$d_{j,0} := \frac{j - 3000}{1000} \qquad \max\left(d^{\langle 0 \rangle}\right) = 3.000$$

$$d_{j,1} := \sigma_{y_i} \cdot \sqrt{1 - \frac{\left(d_{j,0}\right)^2}{\left(\sigma_x\right)^2}}$$

Error Ellipse

$d_{3000,0} = 0.000$ $\quad \mathrm{Re}\left(d_{3000,1}\right) = 0.500$

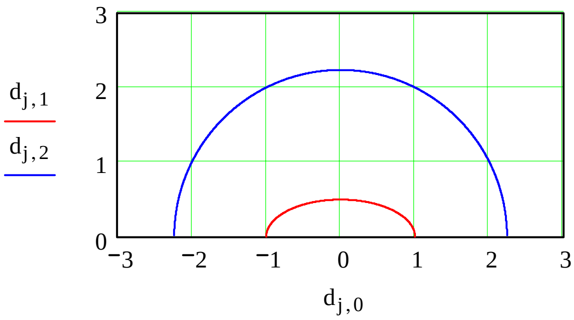
$d_{4000,0} = 1.000$ $\quad \mathrm{Re}\left(d_{4000,1}\right) = 0.000$

$$d_{j,2} := \sqrt{\left[(2 \cdot dRMS)_i\right]^2 - \left(d_{j,0}\right)^2}$$

Integration contour (error circle)

$d_{3000,0} = 0.000$ $\quad \mathrm{Re}\left(d_{3000,2}\right) = 2.236$

$d_{5237,0} = 2.237$ $\quad \mathrm{Re}\left(d_{5237,2}\right) = 0.000$



Probability integral

Integration boundaries

$i := 0..4$

$a_i := (2 \cdot dRMS)_i$

$b_i := (2 \cdot dRMS)_i$

$a_i =$	$b_i =$
2.000	2.000
2.062	2.062
2.236	2.236
2.500	2.500
2.828	2.828

$$P_i := \frac{1}{2 \cdot \pi \cdot \sigma_x \cdot \sigma_{y_i}} \cdot \int_{-a_i}^{a_i} \int_{-\left[\sqrt{(b_i)^2 \cdot \left| 1 - \frac{x^2}{(a_i)^2} \right|}\right]}^{\sqrt{(b_i)^2 \cdot \left| 1 - \frac{x^2}{(a_i)^2} \right|}} e^{\left[\left(-\frac{1}{2}\right) \cdot \left[\frac{x^2}{\sigma_x{}^2} + \frac{y^2}{(\sigma_{y_i})^2}\right]\right]} dy\, dx$$

$k_i =$	$(2dRMS)_i =$	$P_i =$
0.010	2.000	0.955
0.250	2.062	0.959
0.500	2.236	0.970
0.750	2.500	0.979
1.000	2.828	0.982

Result: *the confidence content (or error probability) varies with the ratio of the underlying standard deviations betwen 0.955 and 0.982 The 2dRMS error measure does not possess a fixed probability. For equal standard deviations a probability of 89.2 % applies.*

As the probability content of the 2dRMS error circle varies with the ratio of the underlying SDs, the Rayleigh distribution cannot be used to determine the probability content. The author has developed the following approximation for the probability content as a function of the SD ratio k:

$$P_{a_i} := 0.95435874 + 0.0017921523 \cdot k_i + 0.0895571 \cdot \left(k_i\right)^2 - 0.064296814 \cdot \left(k_i\right)^3$$

$$2dRMS = \begin{pmatrix} 2.000 \\ 2.062 \\ 2.236 \\ 2.500 \\ 2.828 \end{pmatrix} \quad P = \begin{pmatrix} 0.955 \\ 0.959 \\ 0.970 \\ 0.979 \\ 0.982 \end{pmatrix} \quad P_a = \begin{pmatrix} 0.954 \\ 0.959 \\ 0.970 \\ 0.979 \\ 0.981 \end{pmatrix}$$

3a. Integration Contour: Circle CEP50 = h*dRMS

CEP50 is produced iteratively by varying h. As the probability varies with the shape of the error ellipse, an error probability of 50% is produced for all k.

Probability integral

Integration boundaries

$$i := 0..4$$

$$h := \begin{pmatrix} 0.6745 \\ 0.7038 \\ 0.7785 \\ 0.8217 \\ 0.8325 \end{pmatrix}$$ *(factors varied iteratively to produce a probability of 50%)*

$$a_i := (h_i \cdot dRMS_i)$$

$$b_i := (h_i \cdot dRMS_i)$$

$a_i =$	$b_i =$
0.675	0.675
0.725	0.725
0.870	0.870
1.027	1.027
1.177	1.177

$$P_i := \frac{1}{2 \cdot \pi \cdot \sigma_x \cdot \sigma_{y_i}} \cdot \int_{-a_i}^{a_i} \int_{-\left[\sqrt{(b_i)^2 \cdot \left|1 - \frac{x^2}{(a_i)^2}\right|}\right]}^{\sqrt{(b_i)^2 \cdot \left|1 - \frac{x^2}{(a_i)^2}\right|}} e^{\left[\left(-\frac{1}{2}\right) \cdot \left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{(\sigma_{y_i})^2}\right]\right]} \, dy \, dx$$

$h =$	$k_i =$	$dRMS_i =$	$h_i \cdot dRMS_i =$	$P_i =$
0.675	0.010	1.000	0.675	0.5000
0.704	0.250	1.031	0.725	0.5000
0.779	0.500	1.118	0.870	0.5000
0.822	0.750	1.250	1.027	0.5000
0.832	1.000	1.414	1.177	0.5000
			CEP50	

Result: *There is no constant relationship between CEP50 and dRMS. CEP50 can be determined from dRMS for a given ratio of the standard deviations k by multiplying dRMS with the applicable factor h.*

For the case of a circular error distribution (k = 1) - and just only for this case - we can check the result above by applying the Rayleigh CDF.

$$P(R) := 0.50 \qquad \sigma := 1$$

$$R := \sqrt{-2 \cdot \ln(1 - P(R))} \cdot \sigma \qquad R = 1.177$$

The multiplier for dRMS is then $\frac{R}{dRMS_4} = 0.833$ *q.e.d*

3b. Integration Contour: Circle CEP95 = v*CEP50

CEP95 is produced iteratively by varying h. As the probability varies with the shape of the error ellipse, an error probability of 95 % is produced for all k.

Probability integral

Integration boundaries

$$i := 0..4$$

$$h := \begin{pmatrix} 0.6745 \\ 0.7038 \\ 0.7785 \\ 0.8217 \\ 0.8325 \end{pmatrix}$$ *(factors varied iteratively to produce CEP50 from dRMS)*

$$v := \begin{pmatrix} 2.906 \\ 2.725 \\ 2.339 \\ 2.128 \\ 2.079 \end{pmatrix}$$ *(factors varied iteratively to produce CEP95 from CEP50)*

$$a_i := \left(h_i \cdot v_i \cdot dRMS_i\right)$$

$$b_i := \left(h_i \cdot v_i \cdot dRMS_i\right)$$

$a_i =$	$b_i =$
1.960	1.960
1.977	1.977
2.036	2.036
2.186	2.186
2.448	2.448

$$P_i := \frac{1}{2\cdot\pi\cdot\sigma_x\cdot\sigma_{y_i}} \cdot \int_{-a_i}^{a_i} \int_{-\left[\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}\right]}^{\sqrt{(b_i)^2\cdot\left|1-\frac{x^2}{(a_i)^2}\right|}} e^{\left[\left(-\frac{1}{2}\right)\cdot\left[\frac{x^2}{\sigma_x^{\,2}}+\frac{y^2}{\left(\sigma_{y_i}\right)^2}\right]\right]}\,dy\,dx$$

$$v = \begin{pmatrix} 2.906 \\ 2.725 \\ 2.339 \\ 2.128 \\ 2.079 \end{pmatrix}$$

$k_i =$	$dRMS_i =$	$h_i \cdot v_i =$	$P_i =$
0.010	1.000	1.960	0.9500
0.250	1.031	1.918	0.9500
0.500	1.118	1.821	0.9500
0.750	1.250	1.749	0.9500
1.000	1.414	1.731	0.9500

Result: *There is no constant relationship between CEP95 and CEP50. CEP95 can be determined from CEP50 (or from dRMS) for a given ratio of the standard deviations by multiplying CEP50 (dRMS) with the applicable factor v (h*v).*

For the case of a circular error distribution (k = 1) - and just only for this case - we can check the result above by applying the Rayleigh CDF.

$P(R) := 0.95 \qquad \sigma := 1$

$R := \sqrt{-2\cdot\ln(1 - P(R))}\cdot\sigma \qquad R = 2.448$

The multiplier for dRMS is then $\frac{R}{dRMS_4} = 1.731$ *q.e.d*

###